An Interpolation Theorem for Graphical Homomorphisms*

by

Frank Harary, Stephen Hedetniemi, and Geert Prins
Ann Arbor, Michigan

In this paper the concept of a homomorphism of a graph is shown to be very closely related to that of the chromatic number of a graph. On the basis of this observation a number of fundamental results are obtained, which relate these two concepts to each other, and establish an Interpolation Theorem for a particular class of homomorphisms, called complete.

The concept of a contraction of a graph is then shown to enjoy a kind of dual relationship with that of homomorphism. On the basis of this duality a few results are obtained, analogous to those which are obtained for homomorphisms, which lead us directly to the statement of the famous Conjecture of Hadwiger.

1. Notation and definitions. Let G be a graph; let $V = V(G)$, with elements $u, v, w, \ldots$, be the set of points of G; and let $E = E(G)$, elements of which are unordered pairs $uv = [u, v]$ of distinct elements of V, be the set of lines of G. Let K_n be the complete graph on the n points $a_1, a_2, \ldots, a_n$. A graph G'

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is a *subgraph* of G if $V' \subset V$ and $E' \subset E$; in this case G is said to be a *supergraph* of G' .

A set S of points of G is *independent* if no two points in S are adjacent. The *point independence number* $\beta_0(G)$ is the largest number of points in an independent set of G . A point and a line *cover* each other if they are incident. The *point covering number* $\alpha_0(G)$ is the smallest number of points in a set T which covers E .

A *coloring* of a graph G is an assignment of colors to the points of G such that no two adjacent points have the same color. An *n-coloring* of G is a coloring of G which uses n colors, i.e., a function τ from V onto $N = \{1, 2, \ldots, n\}$ such that whenever u, v are adjacent, $u\tau \neq v\tau$.[1] An n-coloring τ is *complete* if for every i, j, $i \neq j$, there exist two adjacent points u, v such that $u\tau = i$ and $v\tau = j$. The *chromatic number* $\chi(G)$ is the smallest number n for which G has an n-coloring. A graph G is *critical* if for every proper subgraph G' of G , $\chi(G') < \chi(G)$; if $\chi(G) = n$ then G is *n-critical*.

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[1] We use here, for conciseness of notation, the convention that $u\tau$ denotes the image of u under the function τ .

2. Homomorphisms. An elementary homomorphism of G is the identification of one pair of non-adjacent points. A homomorphism of G is a sequence of elementary homomorphisms. Thus a homomorphism (cf. Ore [9, p. 83]) of a graph G onto the points of a graph H is a function ϕ from $V(G)$ onto $V(H)$ such that whenever u, v are adjacent in G, $u\phi$, $v\phi$ are adjacent[2] in H. The image of G under the homomorphism ϕ is the graph $G\phi$, where $V(G\phi) = \{u\phi : u \in V(G)\}$ and $E(G\phi) = \{[u\phi, v\phi] : [u, v] \in E(G)\}$.

Two graphs G and H are isomorphic, written $G \cong H$, if there is a $1-1$ correspondence between their point sets which preserves adjacency, i.e., if there exists a homomorphism ϕ from G onto the points of H which is $1-1$, such that $G\phi = H$. A homomorphism ϕ of graph G is complete of order n if $G\phi \cong K_n$.

Theorem 1. For every (complete) n-coloring τ of a graph G there exists a (complete) homomorphism ϕ of G onto K_n,

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[2] Note that it is not specified that every line of H be the "image" under ϕ of a line of G.

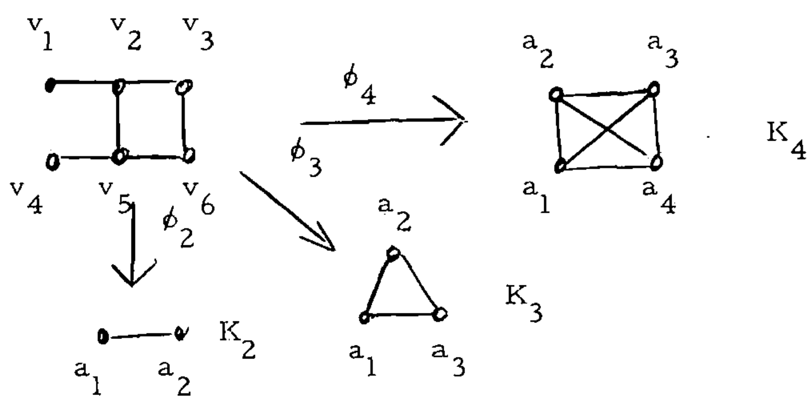
and conversely.

Proof. If $u\tau = i$, let $u\phi = a_i$, and conversely.

The next eight corollaries follow rather directly from this theorem; their proofs are quite simple and are omitted.

Corollary 1.1. (Ore [9, p. 228]) If $\chi(G) = n$ then G has a complete homomorphism of order n.

While this corollary asserts that every graph has at least one complete homomorphism, a given graph may have several complete homomorphisms, of different orders. Figure 1 illustrates this possibility; the graph G has complete homomorphisms of orders 2, 3, and 4.



$\phi_2 : v_1, v_3, v_5 \to a_1$
$\quad v_2, v_4, v_6 \to a_2$

$\phi_3 : v_2, v_6 \to a_1$
$\quad v_1, v_5 \to a_2$
$\quad v_3, v_4 \to a_3$

$\phi_4 : v_2 \to a_1$
$\quad v_5 \to a_2$
$\quad v_1, v_6 \to a_3$
$\quad v_3, v_4 \to a_4$

Figure 1.

Corollary 1.2. The smallest order of all complete homomorphisms of a graph G is $\chi(G)$.

Corollary 1.3. (Hajos)[3] For any homomorphism ϕ of a graph G, $\chi(G) \leq \chi(G\phi)$.

Corollary 1.4. For any graph G having p points and point independence number β_o,

$$p/\beta_o \leq \chi(G) \leq p - \beta_o + 1 .$$

Proof. The right inequality follows from the preceding corollary; the left inequality is noted in Ore [9, p. 225] and Berge [1, p. 37] .

Corollary 1.5. For any graph G and any independent set S of points of G,

$$\chi(G) - 1 \leq \chi(G - S) \leq \chi(G) .$$

[3] See Ringel [10, p. 27]

Corollary 1.6. (Dirac [2, p. 164]) For any critical graph G and any independent set S of points of G,

$$\chi(G) - 1 = \chi(G - S) .$$

Corollary 1.7. For any graph G and any elementary homomorphism ϵ of G,

$$\chi(G) \leq \chi(G\epsilon) \leq \chi(G) + 1 .$$

Corollary 1.8. For any critical graph G and any elementary homomorphism ϵ of G,

$$\chi(G) = \chi(G\epsilon) .$$

Let $\psi(G)$ denote the maximum order of all complete homomorphisms of G. While Corollary 1.2 asserts that $\chi(G)$ is the minimum order of all complete homomorphisms of G, it remains an open question to determine a decent description of $\psi(G)$. The following result, which extends Corollary 1.4, establishes a bound for $\psi(G)$.

<u>Theorem 2.</u> For any graph G having p points and point independence number β_o

$$\chi(G) \leq \psi(G) \leq p - \beta_o + 1 .$$

<u>Proof.</u> Let $\psi(G) = t$, and let ϕ be a complete homomorphism of G onto K_t. Consider the partition of $V(G)$ into sets $V_1, V_2, \ldots, V_t$, such that $u \in V_i$ if and only if $u\phi = a_i$. Let S be any independent set of G containing β_o points, and consider where among the sets V_i the points in S lie. Three possibilities exist for any set V_i : (1) V_i contains no points of S, (2) V_i contains some points of S and some points of $V - S$, or (3) V_i contains only points of S. Note however that at most one set V_i can contain only points

of S. It follows therefore that $t - 1$ of these sets contain at least one point which is not a point of S, i.e.,

$$p - (t - 1) \geq \beta_o \text{ or ,}$$

$$\psi(G) \leq p - \beta_o + 1 .$$

Corollary 2.1. For any graph G having point covering number α_o,

$$\chi(G) \leq \psi(G) \leq \alpha_o + 1 .$$

Proof. This follows immediately from a result of Gallai [4] which states that $p = \alpha_o + \beta_o$.

Perhaps the most natural bound for $\psi(G)$, where G has q lines, is the largest integer r such that

$$\binom{r}{2} \leq q .$$

As one might expect, however, there are cases in which each of the two bounds, r and $p - \beta_o + 1$, gives a better estimate than the other, as the examples in Figure 2 illustrate.

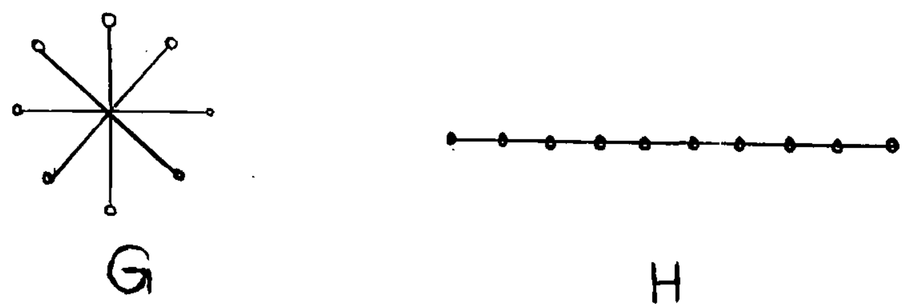


Figure 2 .

For G, $p = 9$, $q = 8$, and $\beta_o = 8$, hence $r = 4$, $p - \beta_o + 1 = 2$, while $\psi(G) = 2$. For H, $p = 10$, $q = 9$, and $\beta_o = 5$, hence $r = 4$, $p - \beta_o + 1 = 6$, while $\psi(G) = 4$.

We are now ready to state our principal result. Without the above observations, in particular Corollary 1.7, the proof of this theorem could be formidable.

Theorem 3. (Homomorphism Interpolation Theorem). For any graph G and any integer n, $\chi(G) \leq n \leq \psi(G)$, G has a complete homomorphism of order n.

Proof. Let $\psi(G) = t$, and consider any complete homomorphism which maps G onto K_t. We know that ϕ can be expressed as a product, say $\epsilon_1, \epsilon_2, \ldots, \epsilon_m$, of elementary homomorphisms. Let $G_1 = G\epsilon_1, G_2 = G_1\epsilon_2, \ldots, G_m = G_{m-1}\epsilon_m = K_t$. By Corollary 1.7 we know that the chromatic number of G_{i+1} is at most one greater than the chromatic number of G_i. It follows therefore that for every n, $\chi(G) \leq n \leq t = \psi(G)$, there exists at least one such graph, say G_i, whose chromatic number is n. But by Corollary 1.1, G_i has a complete homomorphism, say ϕ_i, of order n. Hence G has a complete homomorphism $\epsilon_1, \epsilon_2, \ldots \epsilon_i\phi_i$ of order n.

3. Contractions. An elementary contraction of a graph G is the identification of one pair of adjacent points. A contraction of G is a sequence of elementary contractions. Thus a contraction[4] of a graph G onto a graph H is a function θ from $V(G)$ onto $V(H)$

[4] cf. Ore [9, p. 85], and Dirac [2, p.162]

such that

(i) for every point $u \in V(H)$, the graph $G - (G - \theta^{-1}(u))$ is a connected subgraph of G, and

(ii) for every line $uv \in E(H)$, there is at leastone line in G joining a point of $\theta^{-1}(u)$ with one of $\theta^{-1}(v)$.

Thus a graph G can be contracted onto a graph H if $G = H$ or if H can be obtained from G by shrinking each of a set of disjoint connected subgraphs of G into a single point. If $G = H$, the contraction is said to be trivial.

A contraction θ of a graph G is <u>complete</u> <u>of</u> <u>order</u> <u>n</u> if $G\theta \cong K_n$.

The proofs of the results that follow make ample use of the following observation. Let θ be an elementary contraction of a line uv of a graph G, and let ϵ be an elementary homomorphism of $G - (uv)$ such that $u\epsilon = v\epsilon$; then $G\theta = (G - uv)\epsilon$. Conversely, if ϵ is an elementary homomorphism of a graph H which identifies non-adjacent points u, v, and θ is an elementary contraction of the line uv in the graph $H \cup uv$; then $H\epsilon = (H \cup uv)\theta$.

On the basis of these observations, we can very easily prove the following.

Theorem 4. For every (complete) contraction θ of a graph G there exists a (complete) homomorphism ϕ of a subgraph G' of G such that $G\theta = G'\phi$; and conversely, for every (complete) homomorphism ϕ of a graph G there exists a (complete) contraction θ of a supergraph G'' of G such that $G\phi = G''\theta$.

Theorem 5. For any graph G and any elementary contraction θ of G ,

$$\chi(G) - 1 \leq \chi(G\theta) \leq \chi(G) + 1 .$$

Proof. Let $G\theta$ be obtained from G by contracting the line uv . Consider the graph $G' = G - uv$. Clearly, $\chi(G') = (G)$ or $\chi(G') = \chi(G) - 1$. Consider next the elementary homomorphism ϵ of G' which identifies points u and v . Clearly, $G'\epsilon = G\theta$, and by Corollary 1.7, $\chi(G'\epsilon) = \chi(G')$ or $\chi(G'\epsilon) = \chi(G') + 1$.

Thus, $\chi(G'_\epsilon) = \chi(G\theta)$ can assume only one of the three values, $\chi(G) - 1$, $\chi(G)$, or $\chi(G) + 1$.

Corollary 5.1. If $\chi(G) = n$, then for every k, $1 \leq k < n$, G has a contraction onto a graph H such that $\chi(H) = k$.

Corollary 5.2. A graph G is critical if and only if for every elementary contraction θ of G, $\chi(G\theta) = \chi(G) - 1$.

There is an analogous interpolation theorem for contractions, which is now immediately evident.

Corollary 5.3. (Contraction Interpolation Theorem) If G has a complete contraction of order n, then for every k, $1 \leq k < n$, G has a complete contraction of order k.

Corollary 5.4. Every non-critical graph G has a non-trivial contraction onto a graph H such that $\chi(G) = \chi(H)$.

It should be observed that the contraction in Corollary 5.4 cannot be assumed to be an elementary contraction; the cycle of length four has no elementary contraction onto a graph whose chromatic number is two.

Corollary 5.5. (Dirac [4, p. 50]) Every non-critical graph G has a contraction onto a critical graph H such that $\chi(G) = \chi(H)$.

It was undoubtedly this sequence of observations, Corollaries 5.1 - 5.5, which led Dirac [4, p. 44] to define a graph G to be contraction-critical if for every non-trivial contraction θ of G, $\chi(G\theta) < \chi(G)$; it follows from Corollary 5.5, of course, as was shown by Dirac [4, p. 50], that if G is contraction-critical then G is critical.

In view of Corollary 1.1 and Corollaries 5.1 - 5.5 it is natural to ask: For a given graph G, does G have a complete contraction or order $\chi(G)$? But this question is precisely that which is raised in the celebrated Conjecture of Hadwiger [7] : Every graph G has a complete contraction of order $\chi(G)$.

Since it can be easily shown that a proof of this conjecture for $n = 5$ would imply the truth of the famous Four Color Conjecture, it does not seem likely that Hadwiger's Conjecture will be settled very quickly.

While considerable effort has been directed towards Hadwiger's Conjecture, most of the results that have been obtained, for example those contained in [9, p. 233] , [3], [4], [5], [8], and [11] , provide sufficient conditions for the existence of a complete contraction of a given order, most often of the orders four and five. It has not yet been established however whether the following weaker statement holds.

Conjecture I. Every graph G has a complete contraction of order $\chi(G) - 1$.

One possible means of proving Hadwiger's Conjecture is suggested by the following extension of Corollary 5.1.

Conjecture II. If $\chi(G) = n$, $G \neq K_n$, then for every k , $1 \leq k \leq n$, G has a non-trivial contraction onto a graph H for which $\chi(H) = k$.

Corollary 5.4 asserts that Conjecture II is true for non-critical graphs, while Corollary 5.3 indicates that the conjecture is not necessarily true for critical graphs. The following theorem, which was first proved by Dirac [4] establishes a class of critical graphs for which Conjecture II is true; we are able to supply a proof of this theorem which uses the basic notions and results on homomorphisms and contractions developed in this paper.

Theorem 6. (Dirac [4, p. 50]) Let G be an n-critical graph, $G \neq K_n$, having a point of degree $n - 1$; then G has a nontrivial contraction onto a graph H such that $\chi(H) \geq n$.

Proof. Let point $u \epsilon V(G)$ have degree $n - 1$. Let v, w be two distinct points which are adjacent to u but not to each other. Two such points must exist, or else $G \cong K_n$. Consider the elementary homomorphism ϵ of G which identifies points v and w ; denote the resulting point by v' . From Corollary 1.8, we conclude that $\chi(G\epsilon) = n$, and thus $G\epsilon$ contains an n-critical subgraph, say G' . But since every point of an n-critical graph must have degree greater than or equal to $n - 1$ by Ore [9, p. 230] , and point u has degree $n - 2$ in $G\epsilon$, it follows that $u \notin V(G')$ and thus that $\chi(G\epsilon - uv') = n$. Hence if in $G\epsilon - uv'$ we identify points u and v' by another elementary homomorphism ϵ' , we obtain a graph $H = (G\epsilon - uv')\epsilon'$ such that, by Corollary 1.7 , either $\chi(H) = n$ or $\chi(H) = n + 1$. But H is a contraction of G , proving the theorem.

Incidentally, if Theorem 6 could be proved for n-critical graphs containing at least one point of degree n (instead of $n - 1$), then the Four Color Conjecture would be settled in the affirmative.

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